[3] Lindsay, R. B. and Margenau, H., *Foundations of Physics* (Dover Publications Inc., New York, 1957), pp. 361–374.
[4] MacRobert, T. M., *Functions of a Complex Variable* (Macmillian Co., New York, 1954), 4th ed., p. 330.
[5] Reddick, H. W. and Miller, H. W., *Advanced Mathematics for Engineers* (John Wiley and Sons Inc., New York, 1951), 3rd ed., p. 131.
[6] Sokolnikoff, I. S., *Tensor Analysis* (John Wiley and Sons Inc., New York, 1951), pp. 277–283.
[7] Uspensky, J. V., *Theory of Equations* (McGraw-Hill Book Co. Inc., New York, 1948), p. 92.

Stresses in an Infinite Elastic Slab of Nonhomogeneous Transversely Isotropic Material

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THE object of this paper is to find the displacement and stresses in an infinite slab of transversely isotropic nonhomogeneous material that has a symmetrical distribution of shearing stresses over a circular area on the face $z = 0$, whereas the face $z = b$ is rigidly fixed. The elastic moduli are supposed to vary exponentially with the depth.

The origin is taken on the free surface $z = 0$ of the infinite slab, and the axis of z is drawn into the body at a right angle to this plane. Assuming the axis of z to be the axis of symmetry in a transversely isotropic material, the following stress-strain relations are obtained:[1]

$$\begin{aligned} p_{xx} &= e_{11}e_{xx} + e_{12}e_{yy} + e_{13}e_{zz} \\ p_{yy} &= e_{12}e_{xx} + e_{11}e_{yy} + e_{13}e_{zz} \\ p_{zz} &= e_{13}(e_{xx} + e_{yy}) + e_{33}e_{zz} \\ p_{yz} &= e_{44}e_{yz} \\ p_{zx} &= e_{44}e_{zx} \\ p_{xy} &= [(e_{11} - e_{12})/2]e_{xy} \end{aligned} \tag{1}$$

The equations of equilibrium in absence of body forces are

$$\begin{aligned} \frac{\partial}{\partial x} p_{xx} + \frac{\partial}{\partial y} p_{xy} + \frac{\partial}{\partial z} p_{xz} &= 0 \\ \frac{\partial}{\partial x} p_{xy} + \frac{\partial}{\partial y} p_{yy} + \frac{\partial}{\partial z} p_{yz} &= 0 \\ \frac{\partial}{\partial x} p_{xz} + \frac{\partial}{\partial y} p_{yz} + \frac{\partial}{\partial z} p_{zz} &= 0 \end{aligned} \tag{2}$$

The components of strain are

$$\begin{aligned} &e_{xx} = \partial u/\partial x \qquad e_{yy} = \partial v/\partial y \qquad e_{zz} = \partial w/\partial z \\ &e_{xy} = (\partial u/\partial y) + (\partial v/\partial x) \qquad e_{yz} = (\partial w/\partial y) + (\partial v/\partial z) \\ &e_{zx} = (\partial u/\partial z) + (\partial w/\partial x) \end{aligned} \tag{3}$$

To solve the problem, assume

$$u = -(\partial\phi/\partial y) \qquad v = \partial\phi/\partial x \qquad w = 0 \tag{4}$$

where ϕ is a function of coordinates.

Then the stress components are obtained as

$$\begin{aligned} &p_{xx} = -A(\partial^2\phi/\partial x\partial y) \qquad p_{yz} = G(\partial^2\phi/\partial x\partial z) \\ &p_{yy} = A(\partial^2\phi/\partial x\partial y) \qquad p_{zx} = -G(\partial^2\phi/\partial y\partial z) \\ &p_{zz} = 0 \qquad p_{xy} = (A/2)[(\partial^2\phi/\partial x^2) - (\partial^2\phi/\partial y^2)] \end{aligned} \tag{5}$$

in which $A = e_{11} - e_{12}$ and $G = e_{44}$. Assuming $A = A_0e^{-mz}$ and $G = G_0e^{-mz}$ and using these relations in (5) and (2), one finds that the third equation is satisfied identically and that the other two give

$$\begin{aligned} \frac{\partial}{\partial y}\left[\frac{A_0}{2}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + G_0\left(\frac{\partial^2\phi}{\partial z^2} - m\frac{\partial\phi}{\partial z}\right)\right] &= 0 \\ \frac{\partial}{\partial x}\left[\frac{A_0}{2}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + G_0\left(\frac{\partial^2\phi}{\partial z^2} - m\frac{\partial\phi}{\partial z}\right)\right] &= 0 \end{aligned} \tag{6}$$

These are satisfied if

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k_0^2\left(\frac{\partial^2\phi}{\partial z^2} - m\frac{\partial\phi}{\partial z}\right) = 0 \tag{7}$$

where $k_0^2 = 2G_0/A_0$.

Transforming this equation into cylindrical coordinates (r,θ,ϕ), one obtains for an axially symmetric distribution of stresses

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + k_0^2\left(\frac{\partial^2\phi}{\partial z^2} - m\frac{\partial\phi}{\partial z}\right) = 0 \tag{8}$$

To solve this equation, assume that $\phi = R(r)Z(z)$; then (8) reduces to

$$(d^2R/dr^2) + (1/r)(dR/dr) + \alpha^2R = 0 \tag{9}$$

and

$$k_0^2(d^2Z/dz^2) - mk_0^2(dZ/dz) - \alpha^2Z = 0 \tag{10}$$

Therefore, the solution of (8) is

$$\phi = \int_0^\infty [A_1e^{n_1z} + B_1e^{n_2z}]J_0(\alpha r)d\alpha \tag{11}$$

where $n_{1,2} = [mk_0 \pm (m^2k_0^2 + 4\alpha^2)^{1/2}]/2k_0$.

The components of displacement and stress are obtained as

$$\begin{aligned} u_\theta &= -\int_0^\infty \alpha[A_1e^{n_1z} + B_1e^{n_2z}]J_1(dr)d\alpha \\ p_{\theta z} &= -G_0\int_0^\infty \alpha[A_1n_1e^{(n_1-m)z} + B_1n_2e^{(n_2-m)z}]J_1(\alpha r)d\alpha \\ p_{r\theta} &= \frac{A_0}{2}\left\{\int_0^\infty \alpha^2\,[A_1e^{(n_2-m)z} + B_1e^{(n_2-m)z}]\right\}J_2(\alpha r)d\alpha \end{aligned} \tag{12}$$

To obtain the boundary conditions, assume the shearing stress at the boundary $z = 0$ to be acting within a circular area of a radius a and to be given by the functions

$$\begin{aligned} &u_\theta = 0 \text{ on } z = b \\ &p_{\theta z} = 0 \text{ on } z = 0 \qquad a < r \\ &p_{\theta z} = F(r) = Qr \text{ on } z = 0 \qquad 0 \leqslant r \leqslant a \end{aligned} \tag{13}$$

where Q is a constant.

Table 1

	1	2	3	4	5
r	$0.1a$	$0.2a$	$0.3a$	$0.4a$	$0.5a$
$(u_\theta)_{z=0}G_0/Q$	−0.0008	−0.0015	−0.0024	−0.0031	−0.0038

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[1] Love, A. E. H., *A Treatise on the Mathematical Theory of Elasticity* (Dover Publications, New York, 1927), 4th ed., p. 160.

Applying the boundary conditions in the first and second equations of (12), one obtains

$$A_1 = -\frac{Qa^2}{G_0\alpha}\frac{J_2(\alpha a)e^{(n_2-n_1)b}}{n_1 e^{(n_2-n_1)b} - n_2} \tag{14}$$

and

$$B_1 = -A_1 e^{(n_1-n_2)b}$$

Then the displacement $(u_\theta)_{z=0}$ is given by

$$(u_\theta)_{z=0} = \frac{Qa^2}{G_0}\int_0^\infty \left[\frac{e^{(n_2-n_1)b} - 1}{n_1 e^{(n_2-n_1)b} - n_2}\right] J_2(\alpha a)J_1(\alpha r)d\alpha \tag{15}$$

To get an idea of how the displacement u_θ changes on $z = 0$ for different values of r, take $m = b = 1$ and $a = k_0 = 0.5$. The result is given in Table 1.

Similar Solutions in Boundary Layer Slip Flow

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Some methods of solution for the boundary layer slip-flow problem have been suggested (see, e.g., Refs. 1–3). To check the accuracy of these solutions, one needs to compare them with exact solutions. The purpose of this note is to show how an exact solution can be obtained by means of the similar-solutions technique.

The basic equation of the incompressible boundary layer in terms of Von Mises' nondimensional coordinates can be written as

$$uu_x = U_e U_{ex} + u(uu_y)_y \tag{1}$$

where U_e is the outer velocity, with the following boundary conditions:

$$u_y(x,0) = k \tag{2}$$

$$u(x,\infty) = U_e \tag{3}$$

$$u(0,y) = g(y) \tag{4}$$

Now put $u^2 = f(x)\,F(z)$, with $z = y/h(x)$.

The functions $f(x)$ and $h(x)$ must be found from Eqs. (1) and (2); one obtains

$$f^{1/2} = h = U_e = 1 + Cx$$

To obtain the function $F(z)$, one needs to solve the following equation:

$$F - (zF'/2) = 1 + (F^{1/2}F''/2C) \tag{5}$$

with the boundary conditions

$$F(\infty) = 1 \qquad F'(0) = 2kF^{1/2}(0)$$

Observe that the function $g(y)$ of Eq. (4) is given by $F(y)$. Equation (5) has been solved numerically, and the results are shown in Fig. 1 for $C = 0.5$ and $k = 1$. Now a satisfactory approximate solution is given which is obtained by substituting for $F^{1/2}$ in Eq. (5) a mean value $F_m^{1/2}(0 < F_m < 1)$. One has then

$$F = 1 + A(1 + 2\zeta)(1 - (\pi\zeta)^{-1/2}\exp(-\zeta) \times \{{}_1F_1(1,\tfrac{1}{2}.\zeta) - [1/(1 + 2\zeta)]\})$$

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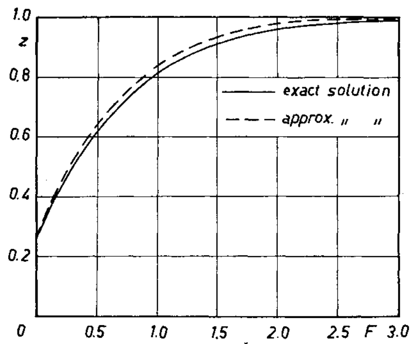

Fig. 1 Exact and approximate velocity function

where ${}_1F_1$ is the confluent hypergeometric function, $\zeta = Cz^2/2F_m^{1/2}$, and the constant A is given by

$$A = \frac{\pi k^2 F_m^{1/2}}{4C} - \frac{\pi k}{4}\left(\frac{k^2F_m}{C^2} + \frac{8F_m^{1/2}}{\pi C}\right)^{1/2}$$

In Fig. 1 is shown this approximate function obtained by assuming for F_m the value 0.5.

References

[1] Hasimoto, H., "Boundary layer slip solutions for a flat plate," J. Aerospace Sci. **25**, 68–69 (1958).
[2] Hassan, H. A., "On skin friction in the slip-flow regime," J. Aerospace Sci. **28**, 335–336 (1961).
[3] Pozzi, A. and Renno, P., "Slip flow with axial pressure gradient," J. Aerospace Sci. **29**, 1393 (1962).

Stress Concentrations around a Small Rigid Spheroidal Inclusion on the Axis of a Transversely Isotropic Cylinder under Torsion

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Introduction

In this note, stresses due to a small rigid inclusion in the form of an oblate spheroid situated on the axis of a large transversely isotropic cylinder under torsion have been found. A corresponding problem for a spherical inclusion in a similar medium was considered by Chatterji.[2] From the results obtained here, stresses due to a rigid inclusion in the form of a prolate spheroid can be deduced by suitable modification.

Solution

The strain-energy function of a transversely isotropic material in cylindrical coordinates is given by

$$W = \tfrac{1}{2}c_{11}(e_{rr}^2 + e_{\theta\theta}^2) + \tfrac{1}{2}c_{33}e_{zz}^2 + c_{13}(e_{rr} + e_{\theta\theta})e_{zz} + c_{12}e_{rr}e_{\theta\theta} + \tfrac{1}{2}c_{44}(e_{\theta z}^2 + e_{rz}^2) + \tfrac{1}{2}c_{66}e_{r\theta}^2$$

where

$$c_{12} = c_{11} - 2c_{66}$$

Considering the large twisted cylinder under torsional stresses

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